The Principle of Least Action and the Euler-Lagrange Equation

Douglas

1 The Problem of Fastest Descent

In June 1696, Johann Bernoulli formally posed this problem as a competition, most likely to show off. Although this was the first proper statement of it, the idea had been considered in the past. The problem is as follows; there are two points, A and B (usually considered in a $2D$ plane, however carrying properties of a $3D$ one), with the second, B lower than the first. The question is simply which path should be taken to minimise the time taken to travel from point A to point B via a constructible ramp without friction (see Fig 1).

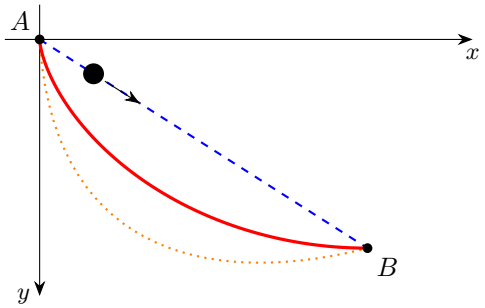


Figure 1: Problem of Fastest Descent.

Before Bernoulli, Galileo had proposed that the answer was simply the arc of a circle because he had proven it was faster than any polygon; despite seeming correct intuitively, Galileo was wrong. Bernoulli having left it for over six months, no one had responded to his challenge. However, Gottfried Leibniz (an old friend of Bernoulli's) persuaded him to postpone the deadline, allowing for foreign entries. This was directed at Newton, who was known to be the best. By this point, Newton was no longer active in science, he had a full time job as the head of the royal mint. He found the letter in the afternoon and had a complete and correct answer by $4am$ the next morning (Bernoulli took over two weeks to complete his). He submitted his solution to the Royal Society without signing it, Johann Bernoulli famously said upon receiving it, "I recognise the lion by his claw" . Although they were both correct, Bernoulli's solution slightly surpassed Newton's, and so it is the one which I will be examining as it is truly brilliant.

Firstly, he looked at how light travels. In the 1st Century AD, this was contemplated by Hero of Alexandria. It could be argued that they are obvious and intuitive ideas, however they did bear much fruit; he said that light follows the shortest physical path from any given point A to any given point B in one medium, *ie* a straight line, and also that the angle of incidence, θ_i is equal to the angle of reflection, θ_r (see Fig 2). Through the years, it was also seen that the sine of the angle intersection, θ_1 into another medium, divided by the sine of the angle of the light which exits in that medium, θ_2 is equal to a constant n which depends on the situation (see Fig 3).

Nobody knew why or how this worked. In 1657, however, Pierre de Fermat had a clever idea. When Hero of Alexandria said that light takes a direct path, he based it on the idea that it was distance being minimised. Fermat thought that it was instead time that was kept to a minimum.

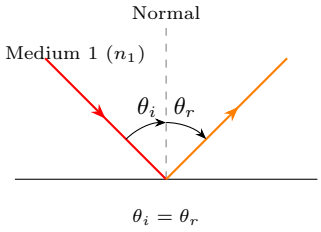


Figure 2: Law of Reflection.

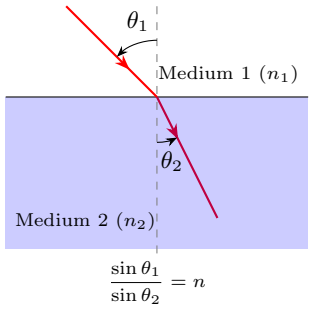


Figure 3: Snell's Law of Refraction.

The only way to prove this was to examine its application to refraction, then look at every single possible path and see that light took the fastest route (see Fig 4). A good example of this would be that person A on the beach sees person B drowning in the water: A can't take a straight journey because A can run faster than swim, however A can't run too far because it is inefficient, meaning A must choose some journey between them. This was a very hard task and caused Fermat to give up for almost five years, after which he eventually came back to it and was successful.

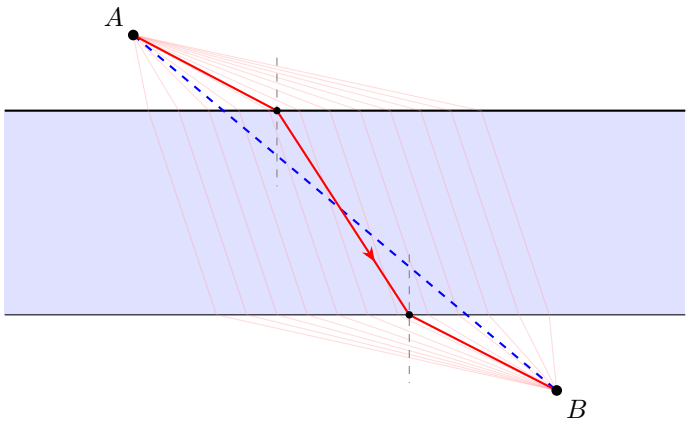


Figure 4: The different paths available for light to take through a medium

Fermat couldn't prove his hypothesis of least time, which he considered a postulate for his proof. Firstly, he set up a system which allowed him to map every path from point A in the air to point B in the water tank which had a boundary line where it changed from the air (the first medium) to the tank (the second medium). He set out the following pieces of information;

- a is the vertical distance from point A to the boundary line
- b is the vertical distance from point B to the boundary line
- L is the fixed horizontal distance between A and B
- x is the variable position where the light ray may strike the boundary

Using Pythagoras' Theorem, the distance which light travels in air is,

$$\sqrt{a^2 + x^2}$$

And in water,

$$\sqrt{b^2 + (L - x)^2}$$

Since time, T is equal to distance divided by velocity, Fermat wrote T as a function of x,

$$T(x) = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (L - x)^2}}{v_2}$$

This is where v_1 and v_2 are the velocities of light in the first and second media respectively. As one would find the minimum of a parabola, Fermat formed an equation where the derivative of T with respect to x is equal to zero,

$$\frac{dT}{dx} = 0$$

Using the chain rule for the rest of the equation gives,

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{L - x}{v_2\sqrt{b^2 + (L - x)^2}} = 0$$

Rearranging gives,

$$\frac{x}{v_1\sqrt{a^2 + x^2}} = \frac{L - x}{v_2\sqrt{b^2 + (L - x)^2}}$$

He noticed that these ratios were equal to those of a right-angled triangle ($\frac{O}{H}$), thus,

$$\frac{x}{\sqrt{a^2 + x^2}} = \sin\theta_1$$
$$\frac{L - x}{\sqrt{b^2 + (L - x)^2}} = \sin\theta_2$$

Substituting this back into his calculus result gave,

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \implies \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

or,

$$\frac{\sin\theta_1}{\sin\theta_2} = n$$

Which not only gives us Snell's law, but also tells us about the constant, n. However, it is most commonly written in the first form, $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$, for practical reasons.

Having considered this, Bernoulli decided to approach the problem of fastest descent as one of optics, not mass. He thought about light going through thinner and thinner media, meaning that light speeds up as it passes through them. If there were infinitely many media, light would travel in a smooth curve, with Snell's law used at every change of medium. With this, we can use the way in which light changes its velocity from layer to layer to model a mass. Considering mass again, the law of conservation of energy is obviously applied. Therefore, the mass must trade between kinetic energy, KE (T) and potential energy, (V) specifically gravitational potential energy, GPE. Furthermore, we can derive the following from the law of conservation of energy, measuring y downwards from A as in Fig 1,

$$0 = \frac{1}{2}m\Delta(v^2) - mg\Delta y$$

Which can be rearranged to,

$$\frac{1}{2}m\Delta(v^2) = mg\Delta y$$

Giving us the relationship at any point of,

$$\frac{1}{2}\Delta(v^2) = g\Delta y$$

Then, since the mass starts at rest at A, where $y = 0$, the total change from A to any point is,

$$v^2 = 2gy \implies v = \sqrt{2gy}$$

Finally, the result of v being proportional to the square root of y is given,

$$v \propto \sqrt{y}$$

By drawing the parallel between the light and the mass, we see that the speed of light is proportional to the distance from the top, y. NB: the speed of light changes and is not c because it is not in a vacuum. Looking at each angle of intersection, θ_i in the different media, using the principle of proportionality which we just deduced, and applying Fermat's version of Snell's law gives us an interesting result,

$$\frac{\sin\theta_1}{\sqrt{y_1}} = \frac{\sin\theta_2}{\sqrt{y_2}} = \frac{\sin\theta_3}{\sqrt{y_3}} = \cdots = \frac{\sin\theta_8}{\sqrt{y_8}} = \dots$$

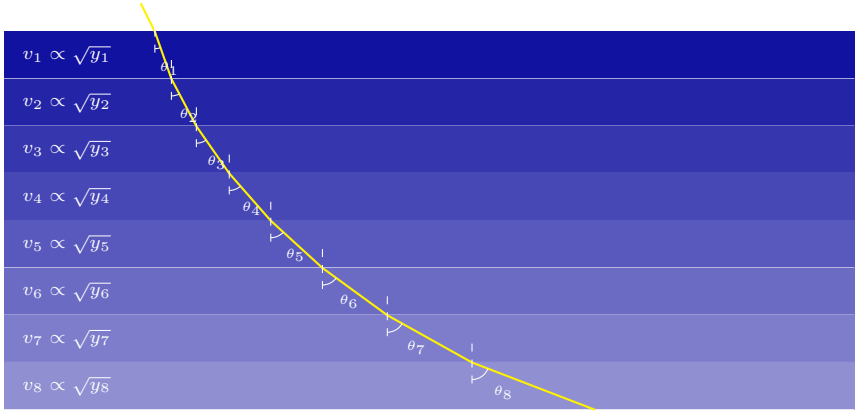


Figure 5: Johann Bernoulli's multi-media refraction showing angular expansion paths

Furthermore, we see that they form the ratio,

$$\frac{\sin\theta_i}{\sqrt{y_i}} = k$$

Where k is a constant. Bernoulli immediately recognised this as the equation of a cycloid. A cycloid is the line drawn by a rolling wheel and is not equal to a circle (see Fig 6). As the curve of fastest descent, it is also called a Brachistochrone curve, after the Greek $\beta\rho\alpha\chi\acute{\iota}\sigma\tau\sigma\varsigma$ meaning 'shortest' mixed with '$\chi\rho\acute{o}\nu\sigma\varsigma$' meaning time. The same curve is also called a Tautochrone curve, from $\tau\alpha\upsilon\tau\acute{o}$ meaning 'the same', although this name comes from a different property. The reason it has the name of 'the same time' is because of an interesting quality which it possesses: that is, a mass dropped from any point along it will always reach point B at the same time as those dropped at different points.

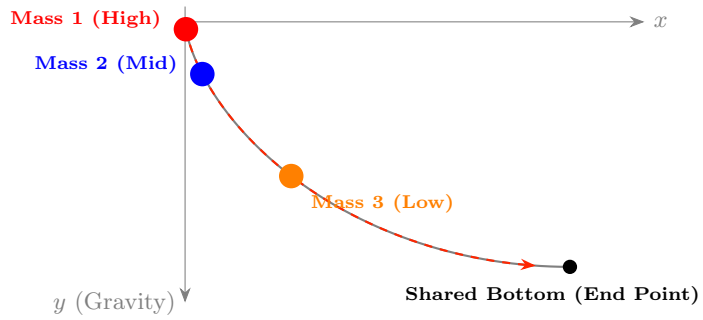


Figure 6: The Tautochrone property of a cycloid: masses 1, 2 and 3 will all reach the bottom at the same time

2 The Principle of Least Action

In a similar way to how Fermat changed Hero of Alexandria's principle of least distance to one of least time, one of Bernoulli's students changed the principle of least time to one of least action around 1740. This student's name was Pierre Louis de Maupertuis. His reason for doing so was because he felt that it was more fundamental. Action, S was defined to be mass, m times velocity, v times distance s.

$$S_0 = mvs$$

Therefore, the farther X goes, the greater the action; the faster X goes, the greater the action; and (if X is a particle or mass) the larger or greater in mass X is, the greater the action. It is quite hard to think about action intuitively, as it isn't a simple quantity like velocity, time or distance If there are multiple parts to a journey, it is just the sum of each of their actions,

$$S_0 = \sum mvs$$

To take an example (see Fig 7), a $1kg$ ball is rolled to a wall at $2ms^{-1}$ from point A, it bounces off and goes to point B at the same speed, where A and B are both $5m$ away from the point of impact of the ball. Since the action is the sum of the actions of the individual parts, it is 1 times 2 times 5 plus 1 times 2 times 5, which is $20kgm^2s^{-1}$.

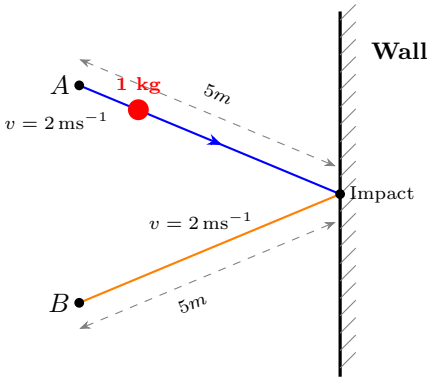


Figure 7: The tracked path of a ball to show its action

Maupertuis claimed that, out of all the possible paths which this ball could've taken to reach point B, this was the one with the least action, and so all objects take the path which minimises their action from A to B. He wrote, "This action is the true expense of nature, which she manages to make as small as possible". Sadly, he was ridiculed and ignored for this idea; Voltaire, once

a friend of his, wrote a thirty-two page document just for the purpose of mocking him: calling him stupid, a plagiarist (specifically of Leibniz's work) and bad at his work. However, this was partly justified, not only because there were rumours that Voltaire's lover had had an affair with Maupertuis, but also because Maupertuis didn't derive or reason the principle, he simply invented it out of nowhere. However, he was supported by one man: that man was Leonhard Euler.

3 Euler's Modifications

The first addition which Euler made was changing the sum into an integral with respect to distance

$$S_0 = \int mvds$$

Doing this meant that he could calculate action whilst factors such as velocity and direction changed continuously. Euler next decided to use it to find the path of a particle around a central mass, like a planet around a star. To find the path which minimised the action, he would have to vary every point along it, of which there are infinitely many.

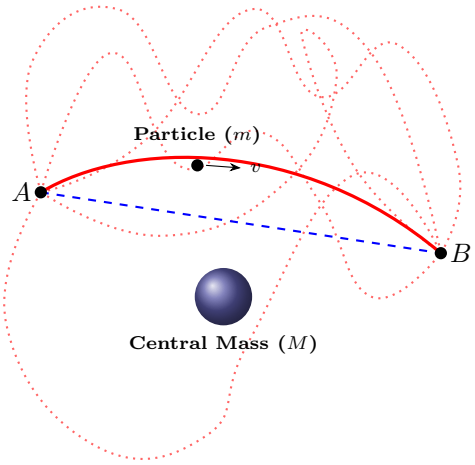


Figure 8: The variable paths around a central mass with infinitely many possible journeys

Seeming to be an impossible task, Euler first had to use conditions to narrow down the possible paths,

- The energy of the mass is conserved throughout the journey
- The energy is the same for all of the paths considered

Maupertuis hadn't considered these before and they were very helpful. In 1755, the 19 year-old Joseph-Louis Lagrange sent a letter to Euler describing the solution of calculus of variations which got rid of the geometric constraints which Euler and Maupertuis were experiencing, eventually leading to a full proof. Their general approach to the problem was as follows.

4 Euler and Lagrange's General Approach

The way they considered finding the path with the least action was similar to that of finding the lowest point of a parabola (just as Fermat did with the refraction of light), which is to set the derivative to zero (see Fig 9). If you move a bit to either side by a first order value, η, the action doesn't change at all, whilst higher orders are negligible,

The Taylor Series expansion of the action with η added is,

$$S[q+\eta] = S[q] + \left(\frac{dS}{dq}\right)\eta + \frac{1}{2}\left(\frac{d^2S}{dq^2}\right)\eta^2 + \dots$$

Since the derivative is zero at the true path,

$$\frac{dS}{dq} = 0$$

We can substitute this into the first order term, which simplifies to,

$$S[q+\eta] = S[q] + \frac{1}{2}\left(\frac{d^2S}{dq^2}\right)\eta^2$$

This means that the first order change completely drops out of the equation. Now, we may subtract the baseline value to show the net change caused by η, which is zero with a first order change and approximately zero (negligible) with higher orders,

$$\delta S = S[q+\eta] - S[q] \approx 0$$

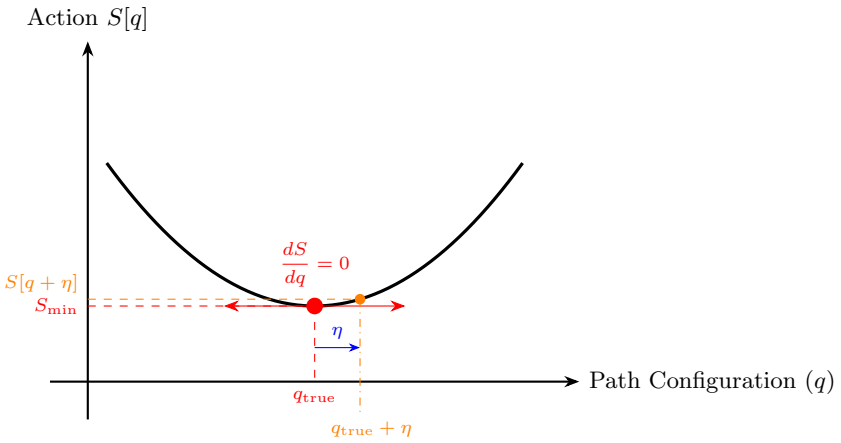


Figure 9: The infinitesimal perturbation, η on a minimum

The rule formed by this tells us that a small deviation from the path of true action doesn't change the overall action,

$$\delta S_0 = 0$$

Strictly speaking, this only tells us that the action is stationary, meaning that it could be a minimum, a maximum or neither, just as a derivative of zero does not always mean a minimum. However, for short enough journeys it is always a minimum, which is why it is still called the principle of least action.

5 The Modern Form of the Equation

The equation $S_0 = \sum mvs$, having been changed to $S_0 = \int mv\,ds$, when combined with $\delta S_0 = 0$ gives,

$$\delta \int mv\,ds = 0$$

Since velocity can be written as $v = \frac{ds}{dt}$, and so $ds = v\,dt$, we may then rewrite the equation as,

$$\delta \int mv^2\,dt = \delta \int 2T\,dt = \delta \int (T+T)\,dt = 0$$

Because $mv^2 = 2KE$, where T is the kinetic energy. Energy is simply kinetic plus potential (just gravitational here is needed), so $E = T + V$, and $T = E - V$ can be substituted in to show,

$$\begin{aligned} 0 &= \delta \int (T + E - V)\, dt \\ &= \delta \int (T - V)\, dt + \delta \int E\, dt \\ &= \delta \int (T - V)\, dt + \delta(Et) \\ &= \delta \int (T - V)\, dt + E\, \delta t + t\, \delta E \end{aligned}$$

But $\delta E = 0$ for different paths, as Euler showed, giving us,

$$\delta \int (T - V)\, dt = -E\, \delta t$$

Since all of these paths have the same energy, they are allowed to take different times. If we instead look at paths with the same travel time, and allow the energy to vary, we can eliminate the final term, $-E\, \delta t = 0$, giving,

$$\delta \int (T - V)\, dt = 0$$

This step is not a full proof, as the two principles compare different sets of paths, however they both lead to exactly the same motion. Note that the two actions themselves are not equal; from the working above, they differ by the energy multiplied by the travel time,

$$\int mv\, ds = \int (T - V)\, dt + Et$$

Therefore, $S = \int (T - V)\, dt$. The final form was first written in this way in 1834 by William Rowan Hamilton, and so it is called Hamilton's Principle; it is the form seen in most textbooks and is what is commonly used nowadays. It has several advantages and differences to Maupertuis' principle,

- It tells us the trajectory of a mass, not just the path of it
- It is an integral over time, not distance: it requires a start and end time as well as a start and end point
- Hamilton's principle allows energy to vary, whilst time is fixed, but Maupertuis' has a fixed energy but lets time vary.

6 An Example to Show Why it Works

An example of this is throwing a ball up in the air, landing at a different point from that from which it was thrown.

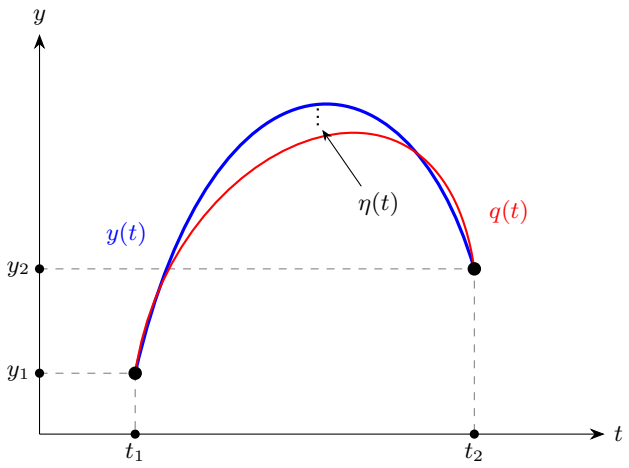


Figure 10: The variation from the true path of a ball thrown up in the air

Where $y(t)$ is the true path, $\eta(t)$ is the variation and $q(t) = y(t) + \eta(t)$ is the variated path, which means that the overall action is,

$$\delta S = S[q(t)] - S[y(t)] = 0$$

We may now solve this equation with energy. The energy of $S[q(t)]$ is its kinetic energy plus its gravitational potential energy. Writing velocity as $\frac{dq}{dt}$, the kinetic energy expands to,

$$\begin{aligned} T &= \frac{1}{2}m\left(\frac{dq}{dt}\right)^2 = \frac{1}{2}m\left(\frac{d}{dt}(y+\eta)\right)^2 \\ &= \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + m\frac{dy}{dt}\frac{d\eta}{dt} + \frac{1}{2}m\left(\frac{d\eta}{dt}\right)^2 \\ &\approx \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + m\frac{dy}{dt}\frac{d\eta}{dt} \end{aligned}$$

Since $\left(\frac{d\eta}{dt}\right)^2$ is negligible. Moreover, its potential energy has the Taylor expansion,

$$V(y+\eta) = V(y) + \eta V'(y) + \frac{1}{2}\eta^2 V''(y) + \cdots \approx V(y) + \eta V'(y)$$

Going back to the original equation, $S = \int_{t_1}^{t_2}(T - V)\,dt$, we can substitute these energy values in and split the integral into two to get,

$$S[q(t)] = \int_{t_1}^{t_2}\left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - V(y)\right)dt + \int_{t_1}^{t_2}\left(m\frac{dy}{dt}\frac{d\eta}{dt} - \eta V'(y)\right)dt$$

For $S[y(t)]$, the kinetic energy is simply $T = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2$ and the potential just $V(y)$, so,

$$S[y(t)] = \int_{t_1}^{t_2}\left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - V(y)\right)dt$$

This is exactly the first integral in $S[q(t)]$, so when we subtract to find the overall change in action, both cancel out to give,

$$\delta S = \int_{t_1}^{t_2}\left(m\frac{dy}{dt}\frac{d\eta}{dt} - \eta V'(y)\right)dt = 0$$

If you integrate the first part by parts, and use the fact that the variation is zero at both ends, $\eta(t_1) = \eta(t_2) = 0$, it gives,

$$\int_{t_1}^{t_2} m\frac{dy}{dt}\frac{d\eta}{dt}\,dt = \left[m\frac{dy}{dt}\eta\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} m\frac{d^2y}{dt^2}\eta\,dt = -\int_{t_1}^{t_2} m\frac{d^2y}{dt^2}\eta\,dt$$

Substituting this back in shows,

$$\delta S = \int_{t_1}^{t_2}\left[-m\frac{d^2y}{dt^2} - V'(y)\right]\eta\,dt = 0$$

And, since η can be anything and the equation is zero, the term inside the square brackets must be zero. Therefore, the path which has stationary action is the one which satisfies the differential equation,

$$-m\frac{d^2y}{dt^2} - V'(y) = 0$$

This can be simplified further. The term $V'(y)$ is just the derivative of the potential energy, which is simply minus the force, $-F(y)$, and the second derivative of height, $\frac{d^2y}{dt^2}$, is simply acceleration, a. Substituting all of these in shows,

$$-ma + F(y) = 0 \implies F = ma$$

Which acts as a proof, since it is in some way derived from a true principle.

7 Applications of the Equation

Looking back, Fermat's and Bernoulli's problems were just special cases of this principle. This law, $\delta S = 0$, can be used to explain almost everything. Through quantum mechanics, it also explains why light and particles take the paths they do, the double slit experiment and many more principles, however this is more to do with physics than mathematics. Having completed this, Euler wrote to Lagrange, "How satisfied would not Mr Maupertuis be, were he still alive, if he could see his principle of action applied to the highest degree of dignity to which it is susceptible". However, there is still the question of why we should use this equation which is very complex compared to the simple equation of Newton's second law. To answer this, we must firstly replace the previous equations and workings with $S = \int_{t_1}^{t_2}(T-V)\,dt$, where the part $L = T-V$ is called the Lagrangian. Furthermore, replacing the previous workings with this would give us the differential equation,

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$$

By inputting kinetic energy and potential energy, it is very easy to solve almost any mechanics problem with this, whilst being compatible in more dimensions simply by solving the equation for more coordinates.

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