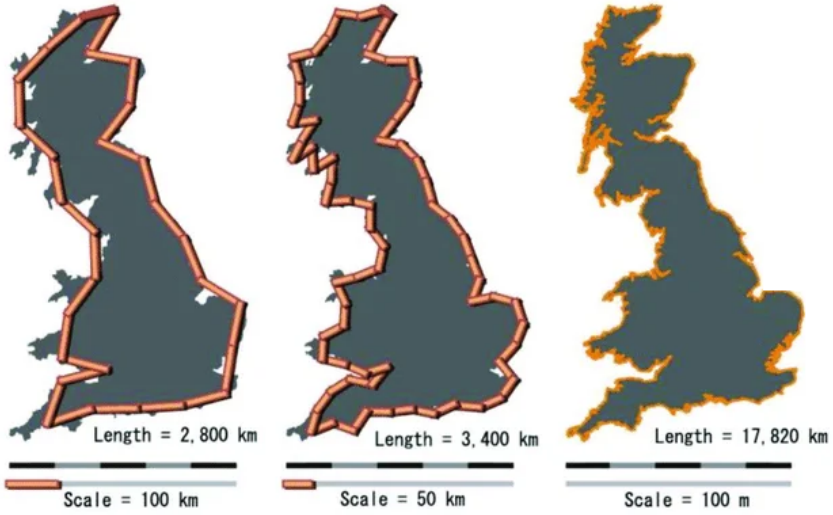
How Long Is a Coastline? The Mathematics of Fractals

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1 The Coastline Paradox

How long is Britain's coastline? The CIA World Factbook records the coastline of the United Kingdom as being 7,723 miles long, while the World Resources Institute gives it as 12,252 miles long. You might wonder how two official sources can differ in a measurement of the same coastline by so much. When you look closer, you are taken into a route of complex mathematics. The reason for this 4,529-mile discrepancy is strangely down to how long the ruler is. How does measuring something more accurately make it longer?



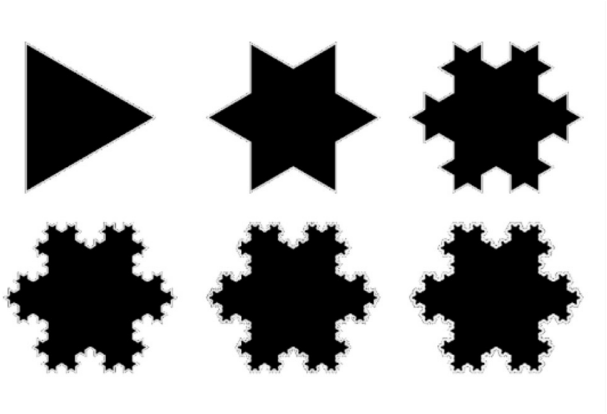
From the diagram above, we can see that if one were to hypothetically line up consecutive 100km rulers and add everything up, the length of Britain's coastline would come to approximately 2,800km. If you did the same with 100m rulers, the length of the coastline is 17,820km, more than 6 times the other estimate. Above, we can see that the 100km ruler cannot measure the coastline with the same precision as the smallest one, skipping over small bays, inlets and bumps. The 100m ruler can measure more of the complex details of the coastline and is able to adjust more to its jagged profile. Since it measures more, the length is longer.

Although this idea works where the coastline is measured with larger scales, it would seem logical to think that there is a fixed maximum length, as there is only so much complexity of a coastline that we can measure. A man named Lewis Fry Richardson thought so too when, in the years before his death in 1953, he was investigating why neighbouring countries such as Spain and Portugal disagreed on the exact length of their shared border. What he discovered surprised him: the shorter the ruler, the longer the measured length, with no sign of it settling down to a fixed value. His findings were published in 1961, and in 1967 the mathematician Benoit Mandelbrot took the idea further: as the length of the ruler approaches zero, the length of the coastline tends to infinity. So, in other words, the perimeter of Britain's coastline depends on the length of the ruler measuring it. The key to understanding why this is lies in grasping the concept of fractals.

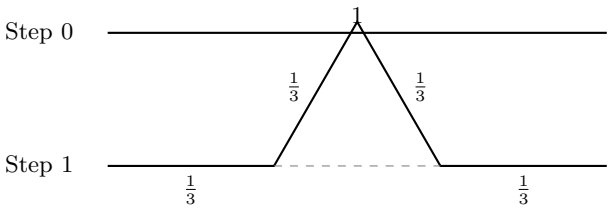
2 Fractals and the Koch Snowflake

The most common definition of a fractal is a pattern that repeats itself at smaller and smaller scales. However, a more precise definition is that a fractal is a shape whose fractal dimension is greater

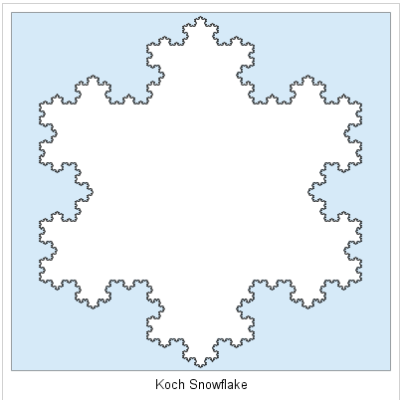
than its ordinary dimension, and this fractal dimension is usually a non-integer (I will explain what this means in due course). Mathematicians do not all agree on a single exact definition. Some fractals have a property called self-similarity, where a small section resembles the original shape. The most famous example is the Koch Snowflake.



As we can see above, this shape is formed by taking an equilateral triangle, splitting each side into 3 equal line segments, and removing the middle one. We then replace each missing line segment with 2 sides of an equilateral triangle, as shown below for a single side.



This process is repeated over and over, resulting in the shape below. This simple rule never changes, turning an ordinary equilateral triangle into an increasingly complex snowflake. This snowflake has some unusual mathematical properties.



To measure the length of this particular fractal, we start with a line segment with length s that follows the rule of the Koch Snowflake. It is 1 side of the equilateral triangle we would ordinarily begin with to construct the Koch Snowflake. The middle third of the line is removed and replaced with 2 other line segments, each with a length of $\frac{s}{3}$. The resulting curve comprises 4 line segments which all have a length of $\frac{s}{3}$. Therefore, from each step to the next, the perimeter is multiplied by $\frac{4}{3}$. Since the snowflake starts as an equilateral triangle with 3 sides, its perimeter after n iterations, P_n, is:

$$P_n = 3s\left(\frac{4}{3}\right)^n$$

where s is the length of one side of the original equilateral triangle and n is the number of iterations. Given that $\frac{4}{3} > 1$, as n becomes larger, the overall perimeter of the Koch Snowflake tends to infinity. With a length that tends to infinity, the Koch Snowflake presents an apparent contradiction: with infinite perimeter, it also has finite area. The reason for this is that, although more triangles are added at each step, they shrink even faster. Every side is replaced by 4 sides, so there are 4 times as many new triangles at each step, but each new triangle has sides one third as long, and so only

$\frac{1}{9}$ of the area of the triangles added before. The total area added therefore shrinks by a factor of $\frac{4}{9}$ every time. If the original triangle has area A_0, the first step adds 3 triangles, each with area $\frac{A_0}{9}$, so the total area is

$$A = A_0 + \frac{A_0}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots\right) = A_0 + \frac{A_0}{3} \cdot \frac{1}{1 - \frac{4}{9}} = \frac{8}{5}A_0$$

Therefore, the area approaches a finite value, exactly $\frac{8}{5}$ of the area of the original triangle, even though the length of the snowflake is infinite. This makes sense in the context of coastlines, as countries do not have infinite area.

3 Fractal Dimension

A mathematician named Benoit Mandelbrot came up with an idea called 'fractal dimension' (D), a way of measuring how complex a shape is as a ratio of the change in detail to the change in scale. It is often a number between 1 and 2 because a fractal line is considered too detailed to be one-dimensional but not detailed enough to be two-dimensional. The equation for fractal dimension of a self-similar fractal is:

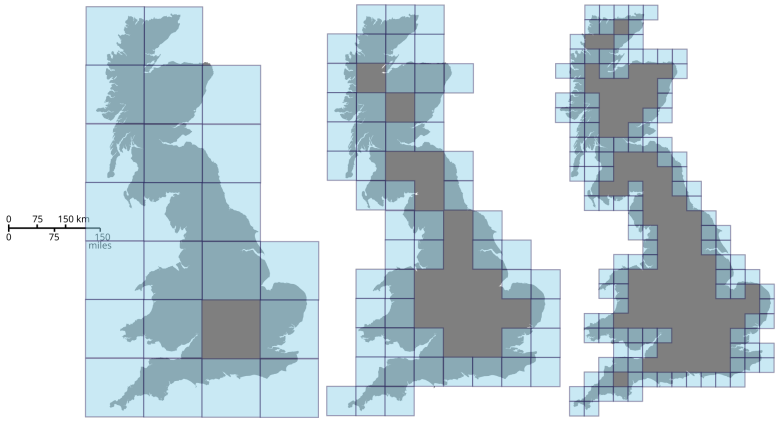
$$D = \frac{\log(N)}{\log(r)}$$

where N = number of smaller self-similar pieces and r = factor by which the object is magnified or scaled down. So, for the Koch Snowflake, N is 4 and r is 3. N is 4 because the original line was split into 4 smaller, equal pieces. r (the magnification factor) is 3 since it tells us how many times you would have to enlarge each small line for it to be the same size as the original. And you now have 4 new segments, making up the new shape. The fractal dimension gives us:

$$D = \frac{\log(4)}{\log(3)} \approx 1.262$$

4 The Fractal Dimension of a Coastline

The equation to calculate the fractal dimension of a coastline is slightly different because coastlines are not perfect, self-similar fractals. Rather, they are fractal-like and so have a different way of calculating the fractal dimension. One way is the box-counting dimension. To calculate it, you cover the fractal with a grid where each box has side length ε and count the number of boxes needed to cover the fractal. ε is then reduced and then $N(\varepsilon)$ is recounted.



This is repeated many times and a graph is plotted, with the vertical axis being $\log N(\varepsilon)$ and the horizontal axis being $\log\left(\frac{1}{\varepsilon}\right)$. A line of best fit is drawn, and the gradient of the line is the fractal dimension D. Using methods like this, the fractal dimension of the west coast of Great Britain has been measured as 1.25, in comparison to Norway, one of the roughest coastlines measured, where $D \approx 1.52$.

A common misconception is that fractals are all self-similar. This is incorrect; Britain's coastline is still a fractal, as what defines a fractal is its fractal dimension being greater than its ordinary dimension, as stated in Section 2. Britain's coastline is a line, so its ordinary dimension is 1, but

its fractal dimension is about 1.25. What makes it a fractal is that the level of roughness remains constant, even when you zoom in closely. In this way, Britain's coastline is statistically self-similar, albeit not a perfect mathematical self-similar fractal like the Koch Snowflake.

5 Fractals in Nature

The concept of fractals is not only seen in coastlines, though. They are found throughout nature, even in something as commonplace as a tree: each branch is like a smaller tree in itself. When you look closer, you see the Y shape appear in all parts of the tree. They also have human-made uses: fractal-shaped antennas can fit a long length of wire into a small space, allowing them to pick up many different frequencies, while doctors can measure the fractal dimension of tissue to help detect cancer, since tumours tend to have more irregular shapes than healthy tissue.



6 Conclusion

Going back to the main question of how long the coastline of Britain is, the truth is, after all this maths, there is no single answer. It is all down to the scale at which we measure it, due to the paradoxical concept of fractals, which can be seen everywhere in nature. They also have countless uses in the modern world and hold promising prospects for the years to come.

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